

Commutative Poisson subalgebras for the Sklyanin bracket and deformations of known integrable models

V. V. Sokolov

Landau Institute for Theoretical Physics,
Kosygina str. 2, Moscow, 117334, Russia

e-mail: sokolov@landau.ac.ru

A. V. Tsiganov

Department of Mathematical and Computational Physics,
St.Petersburg University, St.Petersburg, 198904, Russia

e-mail: tsiganov@mph.phys.spbu.ru

A hierarchy of commutative Poisson subalgebras for the Sklyanin bracket is proposed. Each of the subalgebras provides a complete set of integrals in involution with respect to the Sklyanin bracket. Using different representations of the bracket, we find some integrable models and a separation of variables for them. The models obtained are deformations of known integrable systems like the Goryachev-Chaplygin top, the Toda lattice and the Heisenberg model.

1 Introduction.

Let us consider a 2×2 matrix $T(\lambda)$ which depends polynomially on the parameter λ

$$T(\lambda) = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \quad (1.1)$$

whose entries are polynomials of the form

$$A(\lambda) = \lambda^N + A_{N-1} \lambda^{N-1} + \dots + A_0, \quad B(\lambda) = B_{N-1} \lambda^{N-1} + \dots + B_0, \quad (1.2)$$

$$C(\lambda) = C_{N-1} \lambda^{N-1} + \dots + C_0, \quad D(\lambda) = D_{N-2} \lambda^{N-2} + \dots + D_0.$$

The algebra \mathfrak{A}_N of all functions depending on the coefficients

$$A_0, \dots, A_{N-1}, B_0, \dots, B_{N-1}, C_0, \dots, C_{N-1}, D_0, \dots, D_{N-2}. \quad (1.3)$$

of the matrix $T(\lambda)$ is a Poisson algebra with respect to the so called Sklyanin bracket [2]. The dimension of generic symplectic leaves for this bracket is equal to $2N$.

The main property of the Sklyanin bracket is that the coefficients of the trace of $T(\lambda)$ generate an N -dimensional commutative subalgebra \mathfrak{A}_N^0 in \mathfrak{A}_N . The generators of \mathfrak{A}_N^0 are *linear* polynomials. In this paper we construct different N -dimensional commutative subalgebras \mathfrak{A}_N^M , $M \in \mathbb{N}$, in \mathfrak{A}_N generated by polynomials of higher degrees. These subalgebras can be regarded as deformations of the standard trace subalgebra \mathfrak{A}_N^0 . The generators of \mathfrak{A}_N^M define a set $S_N^M = \{I_1^M, \dots, I_N^M\}$ of N integrals in involution. The separation variables for all these sets S_N^M coincide with those for S_N^0 whereas the separated curves are different.

Using known representations for the Sklyanin bracket and our families S_N^M of integrals in involution we can construct new integrable models. They are deformations of integrable models corresponding to the standard trace integrals. One of the examples is related to known representation [2, 1] of \mathfrak{A}_2 on the Poisson algebra $e(3)$.

The most interesting models are related to the commutative subalgebras \mathfrak{A}_N^1 possessing a quadratic polynomial in variables (1.3) which can be regarded as a Hamiltonian. For example, the subalgebra \mathfrak{A}_2^1 gives us the quadratic deformation of the Goryachev-Chaplygin top found in [7]. In this paper we construct integrable deformations of arbitrary degree for the Goryachev-Chaplygin Hamiltonian.

The well-known Goryachev-Chaplygin case (with the additional gyrostatic term) in rigid body dynamics is described by the following Hamiltonian

$$H = J_1^2 + J_2^2 + 4J_3^2 + 2c_1x_1 + 2c_2x_2 + c_3J_3, \quad (1.4)$$

where c_i are arbitrary constants. The Lie-Poisson brackets

$$\{J_i, J_j\} = \varepsilon_{ijk} J_k, \quad \{J_i, x_j\} = \varepsilon_{ijk} x_k, \quad \{x_i, x_j\} = 0, \quad i, j, k = 1, 2, 3, \quad (1.5)$$

where ε_{ijk} is the standard totally skew-symmetric tensor, defines the corresponding equations of motion. These brackets possess two Casimir elements (x, x) and (x, J) , where $J = (J_1, J_2, J_3)$, $x = (x_1, x_2, x_3)$ and (x, y) stands for the scalar product in \mathbb{R}^3 .

On the fixed level $(x, J) = 0$ of the second Casimir element the Hamiltonian (1.4) commutes with an additional cubic integral of motion. This fact ensures the integrability of the Goryachev-Chaplygin case.

The quadratic deformation of the Goryachev-Chaplygin Hamiltonian given by

$$H = I_1 = J_1^2 + J_2^2 + 4J_3^2 + 2c_1x_1 + 2c_2x_2 + c_3J_3 + 4(a_1x_1 + a_2x_2)J_3 - (a_1^2 + a_2^2)x_3^2 \quad (1.6)$$

also has an additional cubic integral [7]. If $c_1 = c_2 = c_3 = 0$ then (1.6) gives us a new partially integrable case (i.e. integrable on a special level $(x, J) = 0$ of one of the integrals of motion) for the Kirchhoff problem of motion of a rigid body in the ideal fluid. A similar deformation of the Kowalewski top has been considered in [5, 6, 7].

As an application of our general scheme we present below a separation of variables for this model. The canonical separated variables for the Goryachev-Chaplygin top [2] are given by

$$q_{1,2} = J_3 \pm \sqrt{J_1^2 + J_2^2 + J_3^2}, \quad p_{1,2} = \frac{1}{2i} \ln \left(q_{1,2}(ix_1 - x_2) - (iJ_1 - J_2)x_3 \right). \quad (1.7)$$

If we substitute two pairs of variables (1.7) into the following separated equation

$$\lambda^3 + a_0\lambda^2 - I_1\lambda + I_0 = c_0\mu - \frac{b_0\lambda^2(x, x)}{\mu}, \quad \lambda = q_{1,2}, \quad \mu = \exp(2ip_{1,2}), \quad (1.8)$$

and solve the couple of linear equations obtained with respect to $I_1 = H$ and I_0 we immediately derive the Hamilton function

$$H_{gch} = J_1^2 + J_2^2 + 4J_3^2 + 2a_0J_3 - i(c_0 + b_0)x_1 + (c_0 - b_0)x_2. \quad (1.9)$$

and additional integrals of motion for (1.9).

Substituting the same variables (p, q) into another separated equation

$$\lambda^3 + a_0\lambda^2 - I_1\lambda + I_0 = (c_1\lambda + c_0)\mu - \frac{(b_1\lambda + b_0)\lambda^2(x, x)}{\mu}, \quad (1.10)$$

we get an integrable system with the following Hamiltonian

$$H = H_{gch} + \left(c_1(iJ_1 - J_2) + b_1(iJ_1 + J_2) \right) x_3 - 2 \left(c_1(ix_1 - x_2) + b_1(ix_1 + x_2) \right) J_3. \quad (1.11)$$

After the canonical transformation

$$x \rightarrow x, \quad J \rightarrow J + Ux, \quad U = \begin{pmatrix} 0 & 0 & -ic_+ \\ 0 & 0 & -c_- \\ ic_+ & c_- & 0 \end{pmatrix}, \quad c_{\pm} = \frac{b_1 \pm c_1}{2}, \quad (1.12)$$

the latter Hamiltonian becomes

$$\begin{aligned} H = J_1^2 + J_2^2 + 4J_3^2 + 2 \left(i(c_1 + b_1)x_1 - (c_1 - b_1)x_2 + a_0 \right) J_3 + c_1 b_1 x_3^2 \\ - i(b_0 + c_0 - a_0(c_1 + b_1))x_1 + (c_0 - b_0 - a_0(c_1 - b_1))x_2, \end{aligned} \quad (1.13)$$

which coincides with (1.6).

In Section 4 the simplest deformations for the Toda lattice and the Heisenberg model are given. We do not know whether physical applications of such deformed models exist.

Acknowledgments. The authors are grateful to the Newton Institute (Univ. of Cambridge) for its hospitality. The research was partially supported by RFBR grants 99-01-00294 and 99-01-00698, INTAS grants 99-1782 and 99-01459, and EPSRC grant GR K99015.

2 Properties of the Sklyanin bracket

The explicit form of the Sklyanin brackets for the coefficients of the matrix (1.1) can be derived from the following operator definition

$$\{\overset{1}{T}(\lambda), \overset{2}{T}(\mu)\} = [r(\lambda - \mu), \overset{1}{T}(\lambda)\overset{2}{T}(\mu)], \quad (2.1)$$

where we use the standard notations $\overset{1}{T}(\lambda) = T(\lambda) \otimes Id$, $\overset{2}{T}(\mu) = Id \otimes T(\mu)$ and

$$r(\lambda - \mu) = \frac{\eta}{\lambda - \mu} \Pi, \quad \text{where} \quad \Pi = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \eta \in \mathbb{C}. \quad (2.2)$$

Example 1. In the simplest case $N = 2$ relation (2.1) is equivalent to

$$\begin{aligned} \{A_1, A_0\} &= 0, & \{A_1, B_1\} &= \eta B_1, & \{A_1, C_1\} &= -\eta C_1, & \{A_1, B_0\} &= \eta B_0, \\ \{A_1, C_0\} &= -\eta C_0, & \{A_1, D_0\} &= 0, & \{A_0, B_1\} &= \eta B_0, & \{A_0, C_1\} &= -\eta C_0, \\ \{A_0, B_0\} &= \eta(B_0 A_1 - A_0 B_1), & \{A_0, C_0\} &= \eta(A_0 C_1 - C_0 A_1), & & & & \\ \{A_0, D_0\} &= \eta(B_0 C_1 - B_1 C_0), & \{B_1, C_1\} &= \{B_1, B_0\} = 0, & & & & \\ \{B_1, C_0\} &= -\eta D_0, & \{B_1, D_0\} &= 0, & \{B_0, C_1\} &= -\eta D_0, & \{B_0, C_0\} &= -\eta D_0 A_1, \\ \{B_0, D_0\} &= -\eta B_1 D_0, & \{C_1, C_0\} &= \{C_1, D_0\} = 0, & \{C_0, D_0\} &= \eta C_1 D_0. & & \end{aligned} \quad (2.3)$$

It was proven in ([2]) that the coefficients of the determinant

$$d(\lambda) = \det T(\lambda) = A(\lambda)D(\lambda) - B(\lambda)C(\lambda) \quad (2.4)$$

belong to the centre of \mathfrak{A} or, in other words, they are Casimir elements for bracket (2.1). The number of the Casimir functions is $2N - 1$ and therefore we have a $4N - 1$ dimensional Poisson manifold with degenerate Poisson structure (2.1) and $2N$ -dimensional generic symplectic leaves.

To bring the Poisson bracket (2.1) into canonical form, a new set of variables

$$d_0, \dots, d_{2N-2}, \quad q_1, \dots, q_N, \quad p_1, \dots, p_N. \quad (2.5)$$

was proposed in [4]. The variables d_0, \dots, d_{2N-2} are the coefficients of (2.4):

$$d(\lambda) = d_{2N-2} \lambda^{2N-2} + \dots + d_0;$$

and the variables q_i are zeros of the polynomial $A(\lambda)$:

$$A(\lambda) = \prod_{j=1}^N (\lambda - q_j);$$

the variables p_i are defined by

$$p_j = \eta \ln B(q_j).$$

It follows from (2.1) that

$$\{A(\lambda), A(\mu)\} = \{B(\lambda), B(\mu)\} = 0$$

and

$$\{A(\lambda), B(\mu)\} = \frac{\eta}{\lambda - \mu} (A(\lambda)B(\mu) - A(\mu)B(\lambda)).$$

These relations imply $\{q_j, q_k\} = \{p_j, p_k\} = 0$ and $\{p_j, q_k\} = \delta_{jk}$, respectively. As usual if we fix values of the Casimir elements d_j , then the canonically conjugated variables q_i, p_i are symplectic coordinates on the corresponding symplectic leaf.

To express the variables (1.3) in terms of (2.5), one can use the formulae

$$\begin{aligned} A(\lambda) &= \prod_{j=1}^N (\lambda - q_j), & D(\lambda) &= \frac{d(\lambda) + B(\lambda)C(\lambda)}{A(\lambda)}, \\ B(\lambda) &= \sum_{j=1}^N e^{\eta p_j} \prod_{k \neq j} \left(\frac{\lambda - q_k}{q_j - q_k} \right), & C(\lambda) &= \sum_{j=1}^N d(q_j) e^{-\eta p_j} \prod_{k \neq j} \left(\frac{\lambda - q_k}{q_j - q_k} \right). \end{aligned}$$

3 A construction of commutative subalgebras

Let us introduce the matrix

$$\tilde{T}(\lambda) = K(\lambda) T(\lambda), \quad (3.1)$$

where $T(\lambda)$ is given by (1.1),

$$K(\lambda) = \begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & 0 \end{pmatrix} (\lambda), \quad (3.2)$$

whose entries are polynomials of the form

$$\begin{aligned} \mathcal{A}(\lambda) &= \mathcal{A}_M \lambda^M + \mathcal{A}_{M-1} \lambda^{M-1} + \dots + \mathcal{A}_0, \\ \mathcal{B}(\lambda) &= b_M \lambda^M + b_{M-1} \lambda^{M-1} + \dots + b_0, & \mathcal{C}(\lambda) &= c_M \lambda^M + c_{M-1} \lambda^{M-1} + \dots + c_0, \end{aligned}$$

and $b_k, c_k \in \mathbb{C}$ are arbitrary parameters. We require that the trace of the new matrix \tilde{T} has the form

$$\text{trace } \tilde{T}(\lambda) = \sum_{i=N}^{N+M} a_{i-N} \lambda^i + \sum_{k=0}^{N-1} I_k \lambda^k, \quad (3.3)$$

where a_i are arbitrary fixed constant parameters. It easy to see that both unknown coefficients \mathcal{A}_i of the polynomial \mathcal{A} and unknown functions I_k in (3.3) are uniquely defined from (3.3). Moreover, they are polynomials in variables (1.3) such that all the polynomials I_k have the same degree $M + 1$ and the degree of \mathcal{A}_i equals $M - i$.

Example 1 (continuation). If $N = 2$ and $M = 1$ the functions \mathcal{A}_i and I_i are given by

$$\begin{aligned} \mathcal{A}_1 &= a_1, & \mathcal{A}_0 &= a_0 - a_1 A_1 - b_1 C_1 - c_1 B_1, \\ I_1 &= a_1 (A_0 - A_1^2) + b_1 (C_0 - A_1 C_1) + c_1 (B_0 - A_1 B_1) + a_0 A_1 + b_0 C_1 + c_0 B_1, & (3.4) \\ I_0 &= (a_0 - a_1 A_1 - b_1 C_1 - c_1 B_1) A_0 + c_0 B_0 + b_0 C_0. & (3.5) \end{aligned}$$

Theorem 1. Polynomials I_i defined by (3.3) commute with each other with respect to the bracket (2.1).

Proof. The explicit form of condition (3.3) is given by

$$\mathcal{A}(\lambda)A(\lambda) + \mathcal{B}(\lambda)C(\lambda) + \mathcal{C}(\lambda)B(\lambda) = \sum_{i=N}^{N+M} a_{i-M} \lambda^k + \sum_{k=0}^{N-1} I_k \lambda^k.$$

Let us substitute $\lambda = q_j$ into this identity. It follows from the definition $d(\lambda) = AD - BC$ that

$$C(q_j) = -\frac{d(q_j)}{B(q_j)} = -d(q_j) \exp(-\eta p_j).$$

Taking this formula into account we get N linear equations

$$\mathcal{C}(q_j) \exp(\eta p_j) - \mathcal{B}(q_j) d(q_j) \exp(-\eta p_j) = \sum_{i=0}^{N-1} I_i q_j^i + \sum_{k=N}^{N+M} a_{k-N} q_j^k, \quad j = 1, \dots, N. \quad (3.6)$$

for N unknowns I_i . Following [8], we can easily show that the functions I_i are in involution with respect to (2.1).

Let us rewrite this system in the matrix form

$$\phi(p, q) = \mathbf{S}(q) I(p, q) + U(q), \quad (3.7)$$

where

$$\phi_j = \mathcal{C}(q_j) \exp(\eta p_j) - \mathcal{B}(q_j) d(q_j) \exp(-\eta p_j), \quad U_j = \sum_k a_k q_j^k$$

and

$$\mathbf{S} = \begin{pmatrix} 1, & q_1, & \cdots & q_1^{N-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1, & q_N, & \cdots & q_N^{N-1} \end{pmatrix}$$

The solution of (3.7) is given by

$$I(p, q) = \mathbf{W}(q) \left(\phi(p, q) - U(q) \right), \quad \mathbf{W}(q) = \mathbf{S}^{-1}(q). \quad (3.8)$$

The matrix \mathbf{S} belongs to the class of so-called Stäckel matrices [8] for which the entries S_{ij} of the i -th row are functions depending on q_i only. Differentiating the identity

$$\sum_i W_{mi} S_{il} = \delta_{ml},$$

with respect to q_j and using the main property of the Stäckel matrices, we have

$$\sum_i \frac{\partial W_{mi}}{\partial q_j} S_{il} = -W_{mj} \frac{\partial S_{jl}}{\partial q_j}.$$

Substituting k for m and eliminating the right hand side $\partial S_{jl}/\partial q_j$ from the two obtained equations, we find

$$\sum_i \left(W_{jk} \frac{\partial W_{im}}{\partial q_j} - W_{jm} \frac{\partial W_{ik}}{\partial q_j} \right) S_{il} = 0.$$

Since the determinant of the matrix \mathbf{S} does not vanish

$$W_{jk} \frac{\partial W_{im}}{\partial q_j} - W_{jm} \frac{\partial W_{ik}}{\partial q_j} = 0 \quad (3.9)$$

for all j, k, i, m .

Calculating the Poisson brackets between I_k and I_m , we obtain

$$\{I_k, I_m\} = \sum_{i,j} \left(W_{jk} \frac{\partial W_{im}}{\partial q_j} - W_{jm} \frac{\partial W_{ik}}{\partial q_j} \right) \frac{\partial \phi_j}{\partial p_j} (\phi_j - U_j(q_j)). \quad (3.10)$$

It follows from (3.9) that $\{I_k, I_m\} = 0$. The proof is complete.

Collorary 1. Variables p_i, q_i are the separated variables for integrals of motion I_j satisfying the separated equations (3.6)-(3.7) whereas variables

$$\lambda_i = q_i, \quad \mu_i = \mathcal{C}(q_i) \exp(\eta p_i)$$

satisfy the characteristic equation $\text{Det}(\tilde{T}(\lambda) - \mu) = 0$. This means that these variables lie on the corresponding equivalent algebraic curves.

Remark 1. Theorem 1 for $M = 0$ is a well-known fact. In the case $N = 2, M = 1$ the matrix \tilde{T} with

$$K(\lambda) = \begin{pmatrix} \lambda + \mathcal{A}_0 & b_0 \\ c_0 & 0 \end{pmatrix}, \quad \mathcal{A}_0 = a_0 - A_1, \quad (3.11)$$

has been proposed in [2] in a non-factorized form. In the paper [9] this matrix was factorized and generalized to the case of arbitrary N . Notice, that our matrix K for $M = 1$ has a more general form

$$K(\lambda) = \begin{pmatrix} \lambda + \mathcal{A}_0 & b_1 \lambda + b_0 \\ c_1 \lambda + c_0 & 0 \end{pmatrix}, \quad \mathcal{A}_0 = a_0 - A_1 - b_1 C_1 - c_1 B_1,$$

than (3.11). Parameters b_1 and c_1 are absolutely essential in constructing new quadratic integrable Hamiltonians.

Remark 2. To generalize the condition (3.3) we can assume that

$$\text{trace } \tilde{T}(\lambda) = \sum_{k \in \hat{\varrho}} a_k \lambda^k + \sum_{i \in \varrho} I_i \lambda^i, \quad a_k \in \mathbb{C}. \quad (3.12)$$

Here $\varrho = \{i_1, \dots, i_N\}$ is an arbitrary subset in the set of numbers $\{0, 1, \dots, N + M\}$ and $\hat{\varrho}$ is the corresponding complement such that $\varrho \cup \hat{\varrho} = \{0, 1, \dots, N + M\}$. But in this case I_i are rational functions of variables (1.3).

Example 1 (continuation). If $N = 2$ and $M = 1$ and

$$\text{trace } \tilde{T}(\lambda) = a_1 \lambda^3 + I_1 \lambda^2 + a_0 \lambda + I_0,$$

these functions are

$$\mathcal{A}_1 = a_1, \quad \mathcal{A}_0 = \frac{1}{A_1} (a_0 - a_1 A_0 - c_0 B_1 - b_0 C_1 - b_1 C_0 - c_1 B_0), \quad (3.13)$$

$$I_1 = \frac{1}{A_1} (a_0 - a_1 (A_0 - A_1^2) - b_1 (C_0 - C_1 A_1) - c_1 (B_0 - B_1 A_1) - c_0 B_1 - b_0 C_1),$$

$$I_0 = \frac{1}{A_1} (A_0 (a_0 - c_1 B_0 - a_1 A_0 - b_1 C_0) + b_0 (C_0 A_1 - A_0 C_1) + c_0 (B_0 A_1 - A_0 B_1)).$$

Remark 3. Theorem 1 can be proven in the same way for matrices (1.1) of slightly different structure. For example, we can assume that the entry $D(\lambda)$ of the matrix $T(\lambda)$ has the form

$$D(\lambda) = \lambda^N + D_{N-1}\lambda^{N-1} + D_{N-2}\lambda^{N-2} + \cdots + D_0.$$

In this case we have one additional variable D_{N-1} and one additional Casimir function $d_{2N-1} = A_{N-1} + D_{N-1}$.

4 Polynomial deformations of known integrable models

If we identify the Sklyanin bracket (2.1) with a fixed Poisson bracket on some phase space \mathcal{M} , then (2.1) can be regarded as an *equation* for matrix $T(\lambda)$ of the form (1.1) whose entries (1.2) are polynomials in λ with coefficients being functions on \mathcal{M} . Such a matrix $T(\lambda)$ defines a representation of (2.1) on \mathcal{M} .

Using known representations and Theorem 1, we can produce hierarchies of deformations for several integrable models such as the Goryachev-Chaplygin top, the Toda lattice, and the Heisenberg magnetic. According to Corollary 1 integrals of motion for all members of such a hierarchy are separable in the same canonical variables p_i, q_i .

4.1 Deformed Goryachev-Chaplygin top

Let us consider the Lie algebra $e^*(3)$ with a natural Lie-Poisson bracket (1.5). The phase space of the Goryachev -Chaplygin top \mathcal{M} is a union of special non-generic coadjoint orbits (symplectic leaves) of $E(3)$ in $e^*(3)$ specified by the fixed value $I_2 = 0$ of the second Casimir operator.

It was observed in [1] that the functions

$$\begin{aligned} A_1 &= -2J_3, & A_0 &= -J_1^2 - J_2^2 - \frac{\varepsilon}{x_3^2}, & B_1 &= x_2 + ix_1, & B_0 &= -(J_2 + iJ_1)x_3, \\ C_1 &= -x_2 + ix_1, & C_0 &= (J_2 - iJ_1)x_3, & D_0 &= x_3^2 \end{aligned}$$

satisfy (2.3) if the Poisson bracket is given by (1.5), $I_2 = 0$ and $\eta = -2i$. Formulae (3.4) and (3.5) give us two integrals I_1 and I_0 such that $\{I_1, I_0\} = 0$. It is easy to verify that (up to constant factors)

$$\begin{aligned} H_1 = I_1 &= a_1 \left(J_1^2 + J_2^2 + 4J_3^2 + \frac{\varepsilon}{x_3^2} \right) + 2a_0 J_3 + b_0(x_2 - ix_1) - c_0(x_2 + ix_1) \\ &+ b_1(2J_3x_2 - J_2x_3 - 2iJ_3x_1 + iJ_1x_3) - c_1(2J_3x_2 - J_2x_3 + 2iJ_3x_1 - iJ_1x_3) \end{aligned} \quad (4.1)$$

which is equivalent to (1.6). The explicit form of the integral I_0 is given by

$$\begin{aligned} I_0 &= 2a_1(J_1^2 + J_2^2)J_3 + a_0(J_1^2 + J_2^2) + b_0(-J_2 + iJ_1)x_3 + c_0(J_2 + iJ_1)x_3 \\ &+ b_1(J_1^2 + J_2^2)(x_2 - ix_1) - c_1(J_1^2 + J_2^2)(x_2 + ix_1) \\ &+ \varepsilon \frac{2a_1J_3 + a_0 + b_1(x_2 - ix_1) - c_1(x_2 + ix_1)}{x_3^2}. \end{aligned} \quad (4.2)$$

As examples we also present explicitly one polynomial deformation of higher degree and one rational deformation. Thus if $M = 2$ the deformation reads as follows:

$$\begin{aligned} H_2 &= H_1 + 4a_2 J_3 \left((J_1^2 + J_2^2 + 2J_3^2) + \varepsilon x_3^{-2} \right) \\ &+ b_2 \left(2iJ_3 x_3 (J_1 + iJ_2) + (J_1^2 + J_2^2 + 4J_3^2 + \varepsilon x_3^{-2})(x_2 - ix_1) \right) \\ &- c_2 \left((2iJ_3 x_3 (J_1 - iJ_2) + (J_1^2 + J_2^2 + 4J_3^2 + \varepsilon x_3^{-2})(x_2 + ix_1)) \right), \end{aligned}$$

where H_1 is defined by (4.1). In the case $M = 1$ the simplest rational deformation

$$\tilde{H} = \frac{H_1 - a_0(2J_3 - 1)}{2J_3}$$

corresponds to (3.13).

4.2 Deformed Toda lattices

The Sklyanin bracket may be identified [3] with the standard bracket in $\mathcal{M} = \mathbb{R}^{2N}$ with the help of the following ansatz for the matrix $T(\lambda)$:

$$T(\lambda) = L_N L_{N-1} \cdots L_1, \quad L_i(\lambda) = \begin{pmatrix} \lambda - p_i & e^{q_i} \\ -e^{-q_i} & 0 \end{pmatrix} \quad q_{i+N} = q_i. \quad (4.3)$$

Here p_i, q_i are canonical variables in \mathbb{R}^{2N} .

The matrix $T(\lambda)$ obeys the Sklyanin bracket (2.1) with $\eta = 1$ and describes the periodical Toda lattice [3]. The trace of the matrix $T(\lambda)$ produces the integrals of motion. The simplest of them are

$$P = - \sum_{i=1}^N p_i, \quad H = - \sum_{i>j}^N p_i p_j - \sum_{i=1}^N e^{q_{i+1}-q_i}. \quad (4.4)$$

The matrix (4.3) has the necessary polynomial structure (1.2) and we may apply Theorem 1 in order to produce deformations of the periodic Toda lattice, polynomial in momenta. For instance, if $M = 1$ then the integral of the lowest degree in trace $\tilde{T}(\lambda)$ is the following quadratic integral

$$I_{N-1} = -a_0 \sum_{i=1}^N p_i - a_1 \left(\sum_{i>j}^N p_i p_j + \sum_{i=1}^N e^{q_{i+1}-q_i} \right) + e^{q_1} (c_0 + c_1 p_1) - e^{-q_N} (b_0 + b_1 p_N). \quad (4.5)$$

4.3 Deformed spin chain

Let the phase space \mathcal{M} be a direct sum of the algebras $sl^*(2)$ with the following Lie-Poisson bracket

$$\{s_3^i, s_{\pm}^i\} = \pm s_{\pm}^i, \quad \{s_+^i, s_-^i\} = 2s_3^i \quad (4.6)$$

in the each algebra. The Sklyanin bracket with $\eta = 1$ is identified with (4.6) by

$$T(\lambda) = L_N L_{N-1} \cdots L_1, \quad L_i(\lambda) = \begin{pmatrix} \lambda + s_3^i & s_-^i \\ s_+^i & \lambda - s_3^i \end{pmatrix}.$$

It is easy to check that such a product has the polynomial structure described in Remark 3 under the restriction that the Casimir function $d_{2n-1} = A_{N-1} + D_{N+1}$ is equal to zero. Adapting our construction to this case we can construct polynomial deformations for the spin chain. In the simplest case $M = 1$ we obtain the following integrable quadratic Hamiltonian

$$\begin{aligned} I_{N-1} = H &= a_0 S_3 + b_0 S_+ + c_0 S_- + a_1 \left(\sum_{j>i}^N (s_+^i s_-^j + s_3^i s_3^j) - S_3^2 \right) \\ &- 2b_1 \left(S_3 S_+ - \sum_{j>i}^N s_3^i s_+^j + \frac{1}{2} \sum_{i=1}^N s_3^i s_+^i \right) \\ &- 2c_1 \left(S_3 S_- - \sum_{j<i}^N s_3^i s_-^j + \frac{1}{2} \sum_{i=1}^N s_3^i s_-^i \right), \end{aligned}$$

where

$$S_3 = A_{N-1} = \sum s_3^i, \quad S_- = B_{N-1} = \sum s_-^i, \quad S_+ = C_{N-1} = \sum s_+^i.$$

If $b_1 = c_1 = 0$ then H coincides with the general quadratic integral for the standard spin chain.

It is well-known that the Hamiltonians for the Toda lattice and the spin chain derived from the factorised form of operator $T(\lambda)$ have to be transformed to bring them to a local form suitable for applications. We do not know such transformations for the deformed models.

References

- [1] V.B. Kuznetsov and A.V. Tsiganov, A special case of Neumann's system and the Kowalewski-Chaplygin-Goryachev top, *J.Phys.*, v.22, p.L73, 1989.
- [2] E.K. Sklyanin, The Goryachev-Chaplygin top and the method of the inverse scattering problem. Differential geometry, Lie groups and mechanics, VI. *Zap. Nauchn. Sem. LOMI.*, v.133, p.236, 1984.
- [3] E.K. Sklyanin, The quantum Toda chain., Nonlinear equations in classical and quantum field theory (Meudon/Paris, 1983/1984), 196–233, *Lecture Notes in Phys.*, v.226, Springer, Berlin, 1985.
- [4] E.K. Sklyanin, Separation of variables—new trends. Quantum field theory, integrable models and beyond (Kyoto, 1994). *Prog. Theor. Phys. (Suppl)*, v.118, p.35, 1995.

- [5] V.V. Sokolov, A new integrable case for the Kirchhoff equation, *Teor.Math.Phys.*, v.128(2), p.31, 2001.
- [6] V.V. Sokolov, A generalized Kowalevski Hamiltonian and new integrable cases on $e(3)$ and $so(4)$, Preprint *nlin.SI/0110022*, 2001.
- [7] V.V. Sokolov, A.V. Tsiganov, On the Lax pairs for the generalized Kowalewski and Goryachev-Chaplygin tops, *Teor.Math.Phys.*, v.(), p., 2002.
- [8] P. Stäckel, Über die integration der Hamilton-Jacobischen differentialgleichung mittels separation der variabeln, Habilitationsschrift, Halle, 1891.
L.A. Pars, An elementary proof of the Stäckel theorem, *American Math. Monthly*, v.56, p.394, 1949.
- [9] A.V. Tsiganov, The Kowalewski top, a new Lax representation. *J.Math.Phys.*, v.38, p.196, 1997.